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# Minimum time control problems for non autonomous differential equations

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## Abstract

In this paper, we investigate a minimum time problem for controlled non-autonomous differential systems, with a dynamics depending on the final time. The minimal time function associated to this problem does not satisfy the dynamic programming principle. However, we will prove that it is related to a standard front propagation problem via the reachability function. Two simple numerical examples are given to illustrate our approach.

*Key words:* Non-autonomous differential equations, minimum time problem, Hamilton-Jacobi-Bellman equation, reachability set, target problem.

*2000 MSC:* 2008 49J15, 49L20, 93B03

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## 1. Introduction

The main goal of a minimum time problem is to determine the minimal time needed by a controlled system to reach a given target. When the dynamics of the system does not have an explicit time dependence, the problem has been widely studied with different approaches. Here, we focus on the Hamilton-Jacobi-Bellman (HJB) method. Let us recall that this approach is based on Dynamic Programming Principle (DPP) studied by R. Bellman [8]. It leads to a characterization of the minimal time function as a solution of an HJB equation, which appears to be well-posed in the framework of *viscosity solutions* introduced by Crandall and Lions [13, 12]. These tools also allow to perform the numerical analysis of the approximation schemes. We refer to [2, Chapter IV], [3] for theoretical studies. Various numerical methods have been also investigated, such as those based on finite difference schemes [14], ENO and WENO schemes [19, 18], discontinuous Galerkin schemes [17], semi-Lagrangian methods [2, Appendix A] and [15].

An interesting by-product of the HJB approach is the synthesis of the optimal control in feedback form. Once the HJB equation is solved, for any starting point, the reconstruction of the optimal trajectory can be performed in real time.

In this paper, for given  $x \in \mathbb{R}^d$ ,  $T \geq 0$  and control  $\alpha \in \mathcal{A} := L^\infty((0, +\infty); A)$ , we consider the trajectory  $y_x^{\alpha, T}$ , solution of the following non-autonomous system:

$$\begin{cases} \dot{y}(t) = f(T - t, y(t), \alpha(t)), & \text{for } t \in (0, T) \\ y(0) = x. \end{cases} \quad (1.1)$$

Given as target a closed subset  $\mathcal{C}$  of  $\mathbb{R}^d$ , we consider the following minimum time function (this is hereafter referred as problem  $(\mathcal{P}_x)$ ):

$$\mathcal{T}(x) := \inf_{T \in \mathcal{R}(x)} T, \quad (1.2)$$

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where  $\mathcal{R}(x) := \{T \geq 0, \text{ such that } \exists \alpha, y_x^{\alpha, T}(T) \in \mathcal{C}\}$ . It is assumed that  $\mathcal{T}(x) = \infty$  in the case  $\mathcal{R}(x) = \emptyset$ . Let us point out that in this problem the dynamics depends also on the final time. Hence, we keep the relevant information on the time interval in which the state is evolving.

An example of problem involving (1.1)-(1.2) can be found in [9]. In that work, the problem is to steer a heavy space launcher to the GTO orbit with minimal propellant consumption and with a *fixed final mass*. Assuming that the launcher is always at full thrust, minimizing the fuel consumption corresponds to reach the target in minimal time. Moreover, since the mass of ergol is non-increasing with respect to time, it can be expressed as a function of  $T - t$  where  $T$  is the final time, and in the dynamics appears a dependency in terms of  $T - t$  instead of  $t$ .

The main difficulty in problem  $(\mathcal{P}_x)$  comes from the fact that the function  $\mathcal{T}$  *does not* satisfy the DPP (see Remark 2.2). Therefore, the function  $\mathcal{T}$  can not be characterized as a solution of an HJB equation. However, we can prove that it is still connected to a front propagation problem. For this, we introduce the reachability function:

$$\vartheta(x, T) := \min_{\alpha \in \mathcal{A}} \varphi(y_x^{\alpha, T}(T)),$$

where  $\varphi : \mathbb{R}^d \rightarrow \{0, 1\}$  is the function such that  $\varphi(x) = 0$  if  $x \in \mathcal{C}$  and  $\varphi(x) = 1$  otherwise. Then we shall use the fact that  $\mathcal{T}$  can be obtained from  $\vartheta$  by means of the following equality:

$$\mathcal{T}(x) = \min \{T \geq 0, \text{ such that } \vartheta(x, T) = 0\},$$

(see Section 2 for details). Moreover,  $\vartheta$  fulfills the DPP and is characterized by an HJB equation with initial condition  $\vartheta(x, 0) = \varphi(x)$ , for every  $x \in \mathbb{R}^d$ , which can then be used for computations.

This paper is organised as follows. In Section 2 we state the main results. In Section 3 we prove that  $\vartheta$  is a viscosity solution of the HJB equation and we conclude in Section 4 by some numerical illustrative tests.

## 2. Main results

In all the sequel, we consider the admissible set of control variables  $\mathcal{A} := L^\infty(\mathbb{R}^+; A)$ , where  $A$  is a compact convex set of  $\mathbb{R}^m$ , with  $m \geq 1$ . Also, we assume that the following holds:

- (H1) The target  $\mathcal{C}$  is a closed subset of  $\mathbb{R}^d$ .
- (Hf1) The dynamics  $f : \mathbb{R} \times \mathbb{R}^d \times A \rightarrow \mathbb{R}^d$  is a continuous function.
- (Hf2) There exists a constant  $K > 0$  such that

$$|f(t, y, \alpha) - f(t, z, \alpha)| \leq K|y - z| \quad \forall y, z \in \mathbb{R}^d, \alpha \in A, t \in \mathbb{R}.$$

- (Hf3) There exists a constant  $K > 0$  such that

$$|f(t, y, \alpha)| \leq K \quad \forall y \in \mathbb{R}^d, \alpha \in A, t \in \mathbb{R}.$$

- (Hf4) For every  $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ ,  $f(t, x, A)$  is a convex set.

**Remark 2.1.** *It is known that when assumption (Hf4) is not satisfied, the dynamics  $f(t, x, \alpha)$  can be convexified. Moreover, by standard arguments, (Hf2) and (Hf3) can be weakened by respectively assuming only  $|f(t, y, \alpha) - f(t, z, \alpha)| \leq K(t)|y - z|$  and  $|f(t, y, \alpha)| \leq K(t)(1 + |y|)$ , where  $K(t)$  is a continuous function.*

Before studying problem  $(\mathcal{P}_x)$ , we note that by using a change of variable  $t \rightarrow T - t$  we obtain also

$$\mathcal{T}(x) = \min \{T \geq 0, \exists \alpha \in L^\infty((0, T), A), \dot{y} = -f(t, y, \alpha), y(0) \in \mathcal{C}, y(T) = x\}. \quad (2.3)$$

Hence  $\mathcal{T}(x)$  is also the minimal time to reach  $x$  starting from the target  $\mathcal{C}$  and with the dynamics  $b(t, y, \alpha) := -f(t, y, \alpha)$ . In order to keep in this new formulation the relevant information on the time

interval on which the system is evolving, it is necessary to consider another minimal time function depending also on the initial time.

More precisely, for a given initial position  $x \in \mathbb{R}^d$ , an initial time  $\tau \in [0, +\infty)$  and a control  $\alpha \in \mathcal{A} := L^\infty((0, \infty), A)$ , denote by  $z_{x,\tau}^\alpha$  the unique solution of the following state equation :

$$\begin{cases} \dot{z}(t) = b(t, z(t), \alpha(t)) & t \in (\tau, +\infty) \\ z(\tau) = x, \end{cases} \quad (2.4)$$

where the dynamics  $b : \mathbb{R}_+^* \times \mathbb{R}^d \times A \longrightarrow \mathbb{R}^d$  is assumed to be Lipschitz-continuous. For each  $(x, \tau)$  we denote by  $\mathfrak{R}(x, \tau)$  the set of times  $T$  for which there exists a trajectory that reaches the target  $C$  in time  $T$ :

$$\mathfrak{R}(x, \tau) := \{T \geq 0 \mid \exists \alpha : z_{x,\tau}^\alpha(T) \in C\}.$$

The minimum time function is then defined as follows:

$$\theta(x, \tau) := \inf_{T \in \mathfrak{R}(x, \tau)} (T - \tau).$$

By classical arguments, and under standard assumptions on  $b$  ( $b$  satisfies assumptions like (Hf1)-(Hf3)), we can easily check that  $\theta(x, \tau)$  fulfills a DPP and obtain that it is a discontinuous viscosity solution of the following HJB equation:

$$-\frac{\partial \theta}{\partial \tau}(x, \tau) + \sup_{\alpha \in A} \{-p \cdot b(\tau, x, \alpha)\} = 1 \quad (x, \tau) \in \mathbb{R}^d \times \mathbb{R}_+^*, \quad (2.5a)$$

$$\theta(x, \tau) = 0 \quad (x, \tau) \in C \times \mathbb{R}_+^*. \quad (2.5b)$$

The above equation (2.5a) is in a non standard evolutive form, it does not give any information at the initial time  $\tau = 0$  and the information that  $\theta$  vanishes on  $\mathbb{R}_+^* \times C$  can not be used to initiate a numerical process based on time-step iterations. In [21], a numerical scheme based on fast-marching method is proposed. However, this scheme does not work for general dynamics functions (see [21]).

Now, we come back to our problem  $(\mathcal{P}_x)$ . Under assumptions (Hf1)-(Hf4), and by using standard arguments we can prove a Dynamic Programming Principle *inequality*: for  $x \in \mathbb{R}^d$  and  $0 \leq h \leq \mathcal{T}(x)$ , we have

$$\mathcal{T}(x) \geq \inf_{T \in \mathcal{R}(x)} \mathcal{T}(y_x^{\alpha,T}(h)) + h. \quad (2.6)$$

However, in general, the reverse inequality of (2.6) is false, and thus the DPP is false, as illustrated by the following.

**Remark 2.2.** Consider the dynamics  $f(t, x, \alpha) = -2 \max(t - 1, 0) \alpha$  with  $x \in \mathbb{R}$ , the set of control variables  $A := [-1, 1]$ , and the target  $C = \{0\}$ . Let us point out that this example satisfies a controllability property.

The minimal time function satisfies  $\mathcal{T}(x) = 1 + \sqrt{|x|}$  for  $x \neq 0$  and  $\mathcal{T}(0) = 0$ . From any  $x \in \mathbb{R}$ , the optimal trajectory reaches the target on  $[0, \mathcal{T}(x) - 1]$  and then stays in the target during the time interval  $[\mathcal{T}(x) - 1, \mathcal{T}(x)]$ . The jump discontinuity of  $\mathcal{T}$  at  $x = 0$  prevents to have an equality in (2.6) for  $x \neq 0$ , close to zero.

The idea is then to introduce the reachability value function defined as follows: Let  $\varphi : \mathbb{R}^d \rightarrow \{0, 1\}$  be the function such that  $\varphi(x) = 0$  if  $x \in C$  and  $\varphi(x) = 1$  otherwise. For each  $(x, T) \in \mathbb{R}^d \times \mathbb{R}_+^*$  we set:

$$\vartheta(x, T) = \inf_{\alpha \in A} \varphi(y_x^{\alpha,T}(T)), \quad (2.7)$$

$y_x^{\alpha,T}$  being the solution of (1.1). The introduction of  $\vartheta$  is motivated by the following link:

**Theorem 2.3.** Let  $\vartheta$  and  $\mathcal{T}$  be respectively defined in (2.7) and (1.2). Then

$$\mathcal{T}(x) = \min \{T \geq 0 \text{ such that } \vartheta(x, T) = 0\}. \quad (2.8)$$

**Proof.** We have

$$\begin{aligned} T \in \mathcal{R}(x) &\Leftrightarrow \exists \alpha \in \mathcal{A} \text{ such that } y_x^{\alpha, T}(T) \in \mathcal{C} \Leftrightarrow \\ &\Leftrightarrow \exists \alpha \in \mathcal{A} \text{ such that } \varphi(y_x^{\alpha, T}(T)) = 0 \Leftrightarrow \vartheta(x, T) = 0. \end{aligned}$$

Thus  $\mathcal{R}(x) = \{T, \text{ such that } \vartheta(x, T) = 0\}$ , and the thesis follows.  $\square$

It is clear now that to study  $\mathcal{T}(x)$  we can study  $\vartheta(x, T)$ , the latter being more easy to handle because we have the following Dynamic Programming Principle.

**Proposition 2.4. (Dynamic Programming Principle)** *Assume (Hf1) and (Hf2). Fix  $x \in \mathbb{R}^d, T \in \mathbb{R}_+^*$ , let  $\vartheta$  be given by (2.7). For any  $\tau$  such that  $0 \leq \tau \leq T$ , we have*

$$\vartheta(x, T) = \inf_{\alpha \in \mathcal{A}} \vartheta(y_x^{\alpha, T}(\tau), T - \tau). \quad (2.9)$$

**Proof.** We observe that, thanks to (Hf1) and (Hf2), for each control  $\alpha \in \mathcal{A}$  we have  $y_x^{\alpha, T}(T) = y_{y_x^{\alpha, T}(\tau)}^{\tilde{\alpha}, T-\tau}(T - \tau)$ , where  $\tilde{\alpha}(\cdot) = \alpha(\cdot + \tau)$ . Therefore the proof follows by standard arguments.  $\square$

Thanks to the DPP we are able to prove that  $\vartheta$  is the unique bilateral viscosity solution (see Definition 3.1 below) of an HJB equation. The result is the following.

**Theorem 2.5. (Hamilton-Jacobi-Equation)** *Assume (Hf1)-(Hf4). The value function defined in (2.7) is the unique bilateral viscosity solution of*

$$\begin{cases} \frac{\partial v}{\partial t}(x, t) + H(t, x, D_x v(x, t)) = 0 & \text{for } (x, t) \in \mathbb{R}^d \times \mathbb{R}_+^*, \\ v(x, 0) = \varphi(x) & x \in \mathbb{R}^d \end{cases} \quad (2.10)$$

where the Hamiltonian is

$$H(t, x, p) = \sup_{\alpha \in \mathcal{A}} \{-p \cdot f(t, x, \alpha)\}. \quad (2.11)$$

The proof will be given in Section 3 below.

### 3. Proof of Theorem 2.5 (Hamilton-Jacobi-Bellman equation for $\vartheta$ )

This Section is devoted to the proof that the value function  $\vartheta$  defined in (2.7) is the unique bilateral viscosity solution of equation (2.10).

We remark that, since  $\varphi$  is a lower semi-continuous function, we are in the framework of bilateral viscosity solution. This definition was introduced by Barron and Jensen in 1990 ([6]). For further references, see also [4], [2, Chapter V], [3, Chapter 5.3] or [16]. For the sake of completeness we recall here the notion of bilateral viscosity solutions.

**Definition 3.1. (Bilateral viscosity solution).** *A l.s.c. function  $v : \mathbb{R}^d \times \mathbb{R}_+^* \rightarrow \mathbb{R}$  is a bilateral viscosity solution of (2.10) if for any function  $\phi \in C^1(\mathbb{R}^d \times \mathbb{R}_+^*)$ , for which  $v - \phi$  achieves a minimum at point  $(x_0, t_0) \in \mathbb{R}^d \times \mathbb{R}_+^*$ , we have that*

$$\phi_t(x_0, t_0) + H(t_0, x_0, D_x \phi(x_0, t_0)) = 0 \quad (3.12)$$

and  $v(x, 0) = \varphi(x)$  for any  $x \in \mathbb{R}^d$ .

Before proving Theorem 2.5 let us state in the following Lemma the main properties of the Hamiltonian (2.11).

**Lemma 3.2. (Properties of the Hamiltonian)** *Assume (Hf1), (Hf2) and (Hf3). Then the Hamiltonian (2.11) fulfills the following properties:*

- i)  $H : \mathbb{R}_+^* \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  is a continuous function.
- ii) For each  $(t, x) \in \mathbb{R}_+^* \times \mathbb{R}^d$  the function  $p \rightarrow H(t, x, p)$  is convex.

iii) There exist a modulus of continuity  $\omega(\cdot)$  and a constant  $K > 0$  such that

$$|H(t, x, p) - H(s, y, p)| \leq K(|x - y| + \omega(|t - s|)), \quad \forall t, s \in \mathbb{R}_+^*, x, y \in \mathbb{R}^d, p \in \mathbb{R}^d.$$

iv) There exists a constant  $K$  such that

$$|H(t, x, p) - H(t, x, q)| \leq K|p - q| \quad \forall p, q \in \mathbb{R}^d, t \in \mathbb{R}_+^*, x \in \mathbb{R}^d.$$

**Proof.** The continuity of the function  $f$  assumed in (Hf1) ensures us the continuity of  $H$ . Moreover, by Definition (2.11),  $H(t, x, \cdot)$  is the supremum of affine functions therefore is convex. We conclude simply remarking that a direct computation shows that assumptions (Hf2), (Hf3) imply iii) and iv).  $\square$

**Proof of Theorem 2.5.** Let us first remark that thanks to the properties of the Hamiltonian  $H$  we are in the framework studied in the original paper of Barron and Jensen [6] in which a uniqueness result is proved (see [6, Theorem 3.8]). Moreover, by (2.7) the initial condition is fulfilled.

We prove now that our value function  $\vartheta$  is a bilateral viscosity solution of (2.10). Our proof follows the ideas of Barron and Jensen in [7].

Let  $\varphi_n : \mathbb{R} \rightarrow \mathbb{R}$  be a monotone increasing sequence of continuous functions pointwise converging to  $\varphi$ . For each  $n \in \mathbb{N}$  we set  $\vartheta_n(x, T) := \min_{\alpha \in \mathcal{A}} \{\varphi_n(y_x^{\alpha, T}(T))\}$ .

The proof will be divided in three steps. First, we prove that  $\vartheta_n$  is a viscosity solution of (2.10) with initial condition  $\vartheta_n(\cdot, 0) = \varphi_n(\cdot)$ , then we prove that the sequence  $\vartheta_n(x, t)$  converge pointwise to  $\vartheta(x, t)$  and we conclude by applying a stability result with respect to the initial condition.

*Step 1.*  $\vartheta_n$  is a viscosity solution of (2.10) with initial condition  $\vartheta_n(\cdot, 0) = \varphi_n(\cdot)$ . We first remark that by definition we have  $\vartheta_n(x, 0) = \varphi_n(x)$ , thus the initial condition is fulfilled. The proof of  $\vartheta_n$  being a viscosity solution of the equation in (2.10) is completely classical and follows from the application of the DPP, Proposition 2.4. For the sake of completeness we briefly recall the proof of  $\vartheta_n$  being a supersolution. Let  $\phi \in C^1(\mathbb{R}^d \times \mathbb{R}_+^*)$ , for which  $\vartheta_n - \phi$  achieves a minimum at point  $(x_0, t_0) \in \mathbb{R}^d \times \mathbb{R}_+^*$ , we want to prove that

$$\phi_t(x_0, t_0) + H(t_0, x_0, D_x \phi(x_0, t_0)) \geq 0. \quad (3.13)$$

Fix  $(x_0, t_0)$  and  $0 \leq \tau \leq t_0$  by (2.9) in DPP (and (Hf4)), we have that there exists a control  $\bar{\alpha}$  such that  $\vartheta_n(x_0, t_0) = \vartheta_n(y_{x_0, t_0}^{\bar{\alpha}}(\tau), t_0 - \tau)$ . Thus for  $\tau$  small enough and using the regularity of the solution with respect to the initial datum, we have:

$$\phi(x_0, t_0) = \vartheta_n(x_0, t_0) = \vartheta_n(y_{x_0, t_0}^{\bar{\alpha}}(\tau), t_0 - \tau) \geq \phi(y_{x_0, t_0}^{\bar{\alpha}}(\tau), t_0 - \tau). \quad (3.14)$$

Thus

$$\phi(x_0, t_0) - \phi(x_0, t_0 - \tau) - \int_0^\tau D_x \phi(y_{x_0, t_0}^{\bar{\alpha}}(s), t_0 - \tau) \cdot f(t_0 - s, y_{x_0, t_0}^{\bar{\alpha}}(s), \bar{\alpha}(s)) ds \geq 0,$$

therefore, by definition of the Hamiltonian (2.11) and applying assumptions (Hf2) and (Hf3), we have

$$\begin{aligned} & \phi(x_0, t_0) - \phi(x_0, t_0 - \tau) + H(t_0, x_0, D_x \phi(x_0, t_0 - \tau))\tau + \\ & + K \int_0^\tau |D_x \phi(y_{x_0, t_0}^{\bar{\alpha}}(s), t_0 - \tau)| |y_{x_0, t_0}^{\bar{\alpha}}(s) - x_0| ds + \\ & + K \int_0^\tau |D_x \phi(y_{x_0, t_0}^{\bar{\alpha}}(s), t_0 - \tau) - D_x \phi(x_0, t_0 - \tau)| ds \geq 0, \end{aligned}$$

and (3.13) follows dividing by  $\tau$  and letting  $\tau \rightarrow 0$ .

*Step 2.*  $\vartheta_n(x, t)$  converge pointwise to  $\vartheta(x, t)$ . Since the sequence  $\varphi_n \uparrow \varphi$ , the sequence  $\vartheta_n$  is monotone increasing too, and  $\vartheta_n \leq \vartheta$ . Therefore

$$\lim_{n \rightarrow \infty} \vartheta_n(x, t) \leq \vartheta(x, t), \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}_+^*. \quad (3.15)$$

Since  $\vartheta_n(x, t) := \min_{\alpha \in \mathcal{A}} \{\varphi_n(y_x^{\alpha, t}(t))\}$ , for each  $n$  and  $(x, t)$  fixed, there exists an optimal control  $\bar{\alpha}_n \in \mathcal{A}$  such that  $\vartheta_n(x, t) = \varphi_n(y_x^{\bar{\alpha}_n, t}(t))$ . The convexity assumption (Hf4) allows us to apply the general

Convergence Theorem, Theorem 1 page 60 in [1]. There exists then a subsequence, that we still call  $(\bar{\alpha}_n)_{n \in \mathbb{N}}$ , and a corresponding sequence of trajectories  $(y_x^{\bar{\alpha}_n, t}(\cdot))_{n \in \mathbb{N}}$  such that

$$\bar{\alpha}_n \rightarrow \bar{\alpha} \text{ weakly}^* L^\infty(\mathbb{R}_+^*; A) \quad \text{and} \quad y_x^{\bar{\alpha}_n, t}(\cdot) \rightarrow y_x^{\bar{\alpha}, t}(\cdot) \text{ uniformly in } [0, t]. \quad (3.16)$$

(For a similar use of this general result see, for instance, [20].) Therefore, fix  $\varepsilon > 0$  there exists a  $N_\varepsilon$  such that  $\forall n \geq N_\varepsilon$  we have

$$\vartheta_n(x, t) = \varphi_n(y_x^{\bar{\alpha}_n, t}(t)) \geq \varphi_{N_\varepsilon}(y_x^{\bar{\alpha}_n, t}(t)) \geq \varphi_{N_\varepsilon}(y_x^{\bar{\alpha}, t}(t)) - \varepsilon \geq \varphi(y_x^{\bar{\alpha}, t}(t)) - \varepsilon \geq \vartheta(x, t) - \varepsilon$$

where we also used:  $(\varphi_n)_{n \in \mathbb{N}}$  is monotone increasing,  $\varphi_n \uparrow \varphi$  and the definition of  $\vartheta$ .

Thus,  $\liminf_{n \rightarrow \infty} \vartheta_n(x, t) \geq \vartheta(x, t)$  for all  $(x, t) \in \mathbb{R}^d \times \mathbb{R}_+^*$ , and the thesis follows also from (3.15).

*Conclusion.* The conclusion follows from the stability with respect to initial condition of the bilateral solution. (See [5, Theorem 17] or [7, Theorem 3].)  $\square$

#### 4. Numerical examples

We consider two simple academic examples for non-autonomous minimum time problems. We solve the HJB equation (2.10), which gives the reachability function  $\vartheta$ , and then use (2.8) to get  $\mathcal{T}$  from  $\vartheta$ . In practice we solve the HJB equation in some time interval  $[0, T_{max}]$  and on some bounded square box  $\Omega = [X_{1,min}, X_{1,max}] \times [X_{2,min}, X_{2,max}]$ . We consider a uniform grid mesh  $(x_i^1, x_j^2)$  using  $N_{x_1} \times N_{x_2}$  points. The time discretization  $(t_n)$  we consider will also satisfy  $t_{n+1} = t_n + \Delta t_n$  where  $\Delta t_n$  is a variable time step.

Several discretization methods are known for solving equation (2.10) (see the introduction of the present paper for a partial overview). Here, since we shall deal with a discontinuous value function  $\vartheta$ , we choose to use the HJB Ultra-bee scheme as in [11]. This scheme computes an approximation  $V_{i,j}^n$  of  $\frac{1}{|C_{i,j}|} \int_{C_{i,j}} \vartheta((x_1, x_2), t_n) dx_1 dx_2$  for  $n \geq 0$  and  $1 \leq i \leq N_{x_1}$ ,  $1 \leq j \leq N_{x_2}$ , where  $C_{i,j}$  is the cell centered at  $(x_i^1, x_j^2)$ . Computations are done by using the approach developed in [10] (front propagation sparse coding) and take only a few seconds in all cases. The initial data  $\varphi$  we use will have values in  $\{0, 1\}$  and therefore we will have  $V_{i,j}^n \in [0, 1]$  because of  $L^\infty$ -stability properties [11]. The time step  $\Delta t_n$  is adapted in order that a stability condition (a CFL condition of the form  $\Delta t_n \max(\frac{1}{N_{x_1}}, \frac{1}{N_{x_2}}) \leq \text{const}$ ) be satisfied.

A numerical approximation  $\mathcal{T}_{i,j}$  of  $\mathcal{T}(x_i^1, x_j^2)$  is computed as follows. We first set  $\mathcal{T}_{i,j} = \infty$  everywhere (a large numerical value). Then, as the computation of  $V^n$  goes on, for  $n = 0$  up to  $n = N - 1$ , we consider

$$\text{if } \left( V_{i,j}^n > 0, V_{i,j}^{n+1} = 0, \text{ and } \mathcal{T}_{i,j} = \infty \right) \text{ then set } \mathcal{T}_{i,j} := t_{n+1}$$

(a similar procedure is used in [10]). In other words,  $\mathcal{T}_{i,j}$  is the first time when the value of the cell  $(i, j)$  goes to 0, otherwise  $\mathcal{T}_{i,j} = \infty$ .

**Example 1:** The evolution follows the dynamics

$$f(t, x, \alpha) := (1 - \alpha)f_0(t, x) + \alpha f_1(t, x)$$

with  $f_0(t, x) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$  and  $f_1(t, x) = 2\pi t \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$  and  $\alpha \in [0, 1]$ . The target is reduced to one point  $\mathcal{C} = \{(1, 0)^T\}$ . The corresponding initial data is

$$\varphi(x) := \begin{cases} 0 & \text{if } x = (1, 0)^T \\ 1 & \text{otherwise} \end{cases}$$

For numerical computations we have chosen to initialize the scheme values with  $V_{i,j}^0 = 0$  if  $(x_i, y_j) = (0, 0)$  and  $V_{i,j}^0 = 1$  otherwise.

In Figure 1 we show the results obtained with  $N_{x_1} = N_{x_2} = 200$ . Computations are performed in the box  $[-2, 2]^2$ . The region where  $\mathcal{T}_{i,j} > 2$  is not drawn. The total number of time iterations is  $N = 1057$  (CPU time is 4.1 s.).

Furthermore we have plotted an optimal trajectory reconstructed from  $(\mathcal{T}_{i,j})$  and starting from  $x = (0, 1.5)^T$  (we have found  $\mathcal{T}(x) = 1.70$  here)

In this example, we observe that a discontinuity of the minimal time function appears. This is natural because there is no controllability assumptions that would imply continuity (this would not be the case for the minimal time function coming from an eikonal equation for instance).

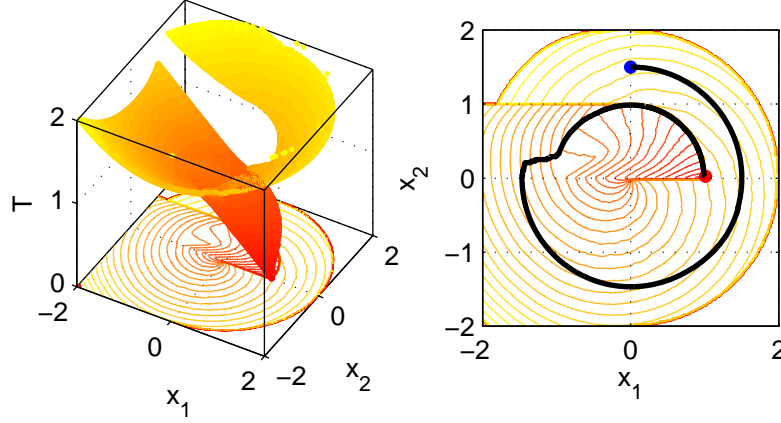


Figure 1: Example 1, 3d view of  $(x_1, x_2, \mathcal{T}(x_1, x_2))$  (left) and isovalues of  $\mathcal{T}$  with an optimal trajectory reconstruction (right).

**Example 2:** The dynamics is

$$f(t, x) := \begin{pmatrix} -2\pi y \\ 2\pi x \end{pmatrix} + \frac{\pi}{2} \sin(\pi t) \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{pmatrix}.$$

(no control dependency). The target  $\mathcal{C}$  is the ball centered at  $(1, 0)$  of radius  $r = 0.25$ . Computations are limited to  $T_{max} = 2$  and to the box  $\Omega = [-3, 3]^2$ . In Figure 2 we show the results obtained with  $N_{x_1} = N_{x_2} = 100$  (CPU time is 0.14 s.).

In this example  $\mathcal{T}$  is again discontinuous but furthermore:

**Remark 4.1.** We observe that the region  $\{\infty > \mathcal{T}(x) > 1.5\}$  is at a non zero distance from the region  $\{\mathcal{T}(x) < 1.5\}$ . Hence  $\mathcal{T}$  does not satisfy a dynamic programming principle.

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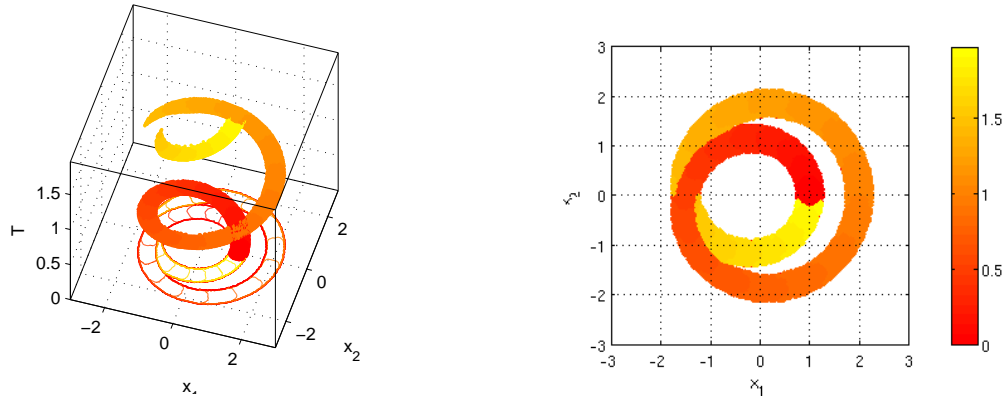


Figure 2: Example 2, Minimal time in 3d view (left) with isovalues plot in the plane  $(x_1, x_2)$ .

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